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A STABILITY ESTIMATE F ORAN ELLIPTIC PROBLEM

BY

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ABSTRACT

We investigate the Caucy problem for linear elliptic operators with C^∞-coefficients at a regular domain $\Omega \subset \mathbb{R}^n$, whic h is a classical example of an ill-posed problem. The Cauchy data are giv en at the manifold $\Gamma \subset \partial\Omega$ and our goal is to obtain a stability estimate in $H_{(4)}(\Omega)$.

1. Introduction and statement of the problem

There are several methods of treating Cauch y problems for second order elliptic operators [6, 7]. In this paper we consider a bounded open region Ω and Γ is a part of $\partial\Omega$ of the class C^m that is open in $\partial\Omega$. Let a function ϕ be in $C^2(\bar{\Omega})$ and we define Ω_ε as $\Omega \cap \{\phi > \varepsilon\}$. With these preliminaries and A an elliptic operator, we define the following Cauch y problem:

$$Au = f \quad \text{on } \Omega, \quad \partial_v^j u = g_j, \quad j \le m-1 \quad \text{on } \Gamma, \qquad \partial^\alpha u \in L^2(\Omega). \tag{1.1}$$

It is well-known that elliptic Cauch y problems are not well-posed in the sense of Hadamard. A famous example given by Hadamard himself [3] shows that we cannot have continuous dependence of the data. Also, existence of solutions for arbitrary Cauchy data $\{g_j\}$ cannot be assured, as shown by a simple argument with the Sch w artz reflection principle [2].

However, extending the Cauch y–Kowalevsky and Holmgren Theorems to the H-con text, it is possible to prove uniqueness of solutions in a weak sense [1, 4].

F or the uniqueness and stability of the problem (1.1), we have the following theorem.

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THEOREM 1.1: *Let ϕ be a function satisfying the pseudoconvexity condition [5]. Then for a solution u to the Cauchy problem (1.1), the following estimate is valid:*

$$\|\partial^\alpha u\|_2(\Omega_\varepsilon) \leq C(F + M^{1-\lambda}F^\lambda),$$

where $F = \|f\|_2(\Omega) + \sum \|g_j\|_{m-j-1/2}(\Gamma)$ and M is the sum of $\|\partial^\alpha u\|_2(\Omega)$.

This theorem guarantees u on Ω, provided we are able to find a strict pseudoconvex function ϕ that agrees with Ω and Γ.

In this study we will be concerned with the solution of the Cauchy problem of the kind

$$A_1 A_2 u = 0 \quad \text{in } \Omega, \quad \partial_v^j u = g_j \quad \text{on } \Gamma, \quad j = 0, \ldots, 3, \tag{1.2}$$

where A_1, A_2 are second-order elliptic operators. In the next section, we will obtain a stability estimate in $H_{(4)}(\Omega)$, for the solution of the problem (1.2).

2. An estimate for the solution of the Cauchy problem

Let us consider the elliptic operator $Au = \sum a_{jk}\partial_j\partial ku + \sum b_j \partial_j u + cu$ whose coefficients satisfy the regularity assumptions [5] and for which a_{jk} are real-valued.

THEOREM 2.1: *For any domain Ω_ε with $\overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$, a solution to the Cauchy problem (1.1) satisfies the following estimate:*

$$\|\partial^\alpha u\|_{(2)}(\Omega_\varepsilon) \leq C(F + \|u\|_2^{1-\lambda}(\Omega)F^\lambda) \quad \text{when } |\alpha| \leq 2,$$

where C and $\lambda \in (0,1)$ depend only on Ω_ε, and $F = \|f\|_2(\Omega) + \|g_0\|_{(2)}(\Gamma) + \|g_1\|_{(1)}(\Gamma)$.

The proof of the theorem can be found in [5]. In the following theorem, we obtain an estimate for the solution of the problem (1.2).

THEOREM 2.2: *Let A_1, A_2 be second-order elliptic operators with $C^2(\bar{\Omega})$-coefficients. Let Ω be a bounded domain in $\mathbb{R}^n$, and Γ a C^4-smooth part of its boundary $\partial\Omega$. Then, an $H_{(4)}(\Omega)$-solution u to the Cauchy problem (1.2) for any subdomain $\Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega \cup \Gamma$ satisfies the following estimate:*

$$\|u\|_{(4)}(\Omega_\varepsilon) \leq C(F + \|u\|_2^{1-\lambda}(\Omega)F^\lambda),$$

where C, λ depend on Ω_ε, $0 < \lambda < 1$ and $F = \|g_0\|_{(4)}(\Gamma) + \cdots + \|g_3\|_{(1)}(\Gamma)$.

Proof: In order to prove this theorem we shall consider the following two Cauchy problems:

$$A_2 u = f, \quad \partial_v^j u = g_j \quad \text{on } \partial\Omega = \Gamma, \quad j = 0, \dots, 3, \tag{2.1}$$

$$A_1 f = 0. \tag{2.2}$$

Here, $f(\Gamma) = A_2 \cup (\Gamma)$ can be estimated by the linear combination of g_0, g_1 and g_2; and $\partial_v f = (A_2 u)_v(\Gamma)$ can be estimated by the linear combination of g_0, g_1, g_2 and g_3.

It is evident that, if u satisfies the original problem, then u must satisfy the problem (2.1) and (2.2).

Since ε in Theorem 2.1 is arbitrary, we can apply the theorem to the Cauchy problem (2.1) and (2.2) and obtain the follwing estimate:

$$\|D^\alpha u\|_2(\Omega_{\varepsilon'}) \leq C(F_1 + \|u\|_2^{1-\lambda_1} F_1^{\lambda_1})(\Omega), \quad |\alpha| \leq 2 \tag{2.3}$$

where

$$F_1 = \|g_0\|_{(2)}(\Gamma) + \|g_1\|_{(1)}(\Gamma) + \|f\|_2(\Omega),$$

and

$$\|D^\alpha f\|_2(\Omega_\varepsilon) \leq C(F_2 + \|f\|_2^{1-\lambda_2} F_2^{\lambda_2})(\Omega_{\varepsilon'}) \quad (\varepsilon' < \varepsilon) \quad \text{and} \quad |\alpha| \leq 2 \tag{2.4}$$

where

$$F_2 = \|A_2 u\|_{(2)}(\Gamma') + \|(A_2 u)_v\|_{(1)}(\Gamma'), \quad \Gamma' = \Gamma \cap \Omega_{\varepsilon'}.$$

Evaluating this equality, we have

$$F_2 \leq C(\|g_0\|_{(4)}(\Gamma) + \cdots + \|g_3\|_{(1)}(\Gamma)).$$

Next, we try to get an estimate for $\|f\|_2(\Omega)$. First of all, by Theorem 2.1 with $\alpha = 0$, we have, for $\forall \varepsilon > 0$,

$$\|f\|_2(\Omega_\varepsilon) \leq C(F_2 + \|f\|_2^{1-\lambda_2} F_2^{\lambda_2})(\Omega). \tag{2.5}$$

Notice that F and f are actually related to $g_0, \dots, g_3$. We may confine ourselves to an arbitrary compact set of $\{g_0, g_1, g_2, g_3\}$. We claim that there is some m_0 such that

$$\frac{F_2}{\|f\|_2} \geq \frac{1}{m_0} \quad \text{for } \forall g_0, \dots, g_3.$$

If this were not the case, i.e. for $\forall m$ and $\exists g_j^m$ $(j = 0, 1, 2, 3)$, then

$$\frac{F_2(g_j^m)}{\|f(g_j^m)\|_2} \leq \frac{1}{m}. \tag{2.6}$$

By our assumption, we may assume $\{g_j^m\}$ itself is convergent to $\{\bar{g}_j\}_{j=0}^3$. Since we obviously have

$$\frac{\|f(\bar{g}_j)\|_2(\Omega_\varepsilon)}{\|f(\bar{g}_j)\|_2(\Omega)} \to 1 \quad (\varepsilon \to 0),$$

there is a neighborhood of $\{\bar{g}_j\}_0^3$, denoted by $N\{\bar{g}_j\}$, such that if $\{g_j\} \in N\{\bar{g}_j\}$ then

$$\frac{\|f(g_j)\|_2(\Omega_{\varepsilon_0})}{\|f(g_j)\|_2(\Omega)} \geq \frac{1}{2}$$

for some ε_0 small enough. On the other hand, with this fixed ε_0, we have by (2.5)

$$\frac{\|f(g_j^m)\|_2(\Omega_{\varepsilon_0})}{\|f(g_j^m)\|_2(\Omega)} \leq C(\varepsilon_0)\Big(\frac{F_2(g_j^m)}{\|f(g_j^m)\|_2} + \Big(\frac{F_2(g_j^m)}{\|f(g_j^m)\|_2}\Big)^{\lambda_2}\Big)$$

and, for m large enough, $\{g_j^m\}_{j=1}^3 \in N\{\bar{g}_j\}$. But the above discussion together with (2.6) gives

$$\frac{1}{2} \leq \frac{\|f(g_j^m)\|_2(\Omega_{\varepsilon_0})}{\|f(g_j^m)\|_2(\Omega)} \leq C(\varepsilon_0)\Big(\frac{F_2(g_j^m)}{\|f(g_j^m)\|_2} + \Big(\frac{F_2(g_j^m)}{\|f(g_j^m)\|_2}\Big)^{\lambda_2}\Big) \to 0 \quad (m \to \infty).$$

This contradiction gives the estimate $\|f\|_2 \leq m_0 F_2$, for some m_0.

Now, it is clear that

$$\|A_2 u\|_2 = \|f\|_2(\Omega_{\varepsilon'}) \leq C \sum_{|\alpha|\leq 2} \|D^\alpha u\|_2 \leq C(F_1 + \|u\|_2^{1-\lambda_1} F_1^{\lambda_1})(\Omega) \tag{2.7}$$

and, by the elliptic condition of the operator, we have

$$\|D^\alpha f\|_2(\Omega) \geq \alpha \|u\|_{(4)}(\Omega_\varepsilon). \tag{2.8}$$

Therefore, with (2.4), (2.7) and (2.8), and using the properties

$$(a+b)^{1-\lambda} \leq a^{1-\lambda} + b^{1-\lambda} \quad \text{and} \quad F_1, F_2, \leq CF,$$

we get

$$\begin{aligned}
\alpha\|u\|_{(4)}(\Omega_\varepsilon) &\leq C(F_2 + \|f\|_2^{1-\lambda_2} F_2^{\lambda_2})(\Omega_{\varepsilon'}) \\
&\leq C(F_2 + (F_1 + \|u\|_2^{1-\lambda_1} F_1^{\lambda_1})^{1-\lambda_2} F_2^{\lambda_2})(\Omega) \\
&\leq C(F_2 + (F_1^{1-\lambda_2} + \|u\|_2^{(1-\lambda_1)(1-\lambda_2)} F_1^{\lambda_1(1-\lambda_2)}) F_2^{\lambda_2})(\Omega) \\
&\leq C(F + (F^{1-\lambda_2} + \|u\|_2^{(1-\lambda_1)(1-\lambda_2)} F^{\lambda_1(1-\lambda_2)}) F^{\lambda_2})(\Omega) \\
&= C(2F + \|u\|_2^{(1-\lambda_1)(1-\lambda_2)} F^{\lambda_1(1-\lambda_2)+\lambda_2}) \\
&\leq C(F + \|u\|_2^{1-\lambda}(\Omega) F^{\lambda}),
\end{aligned}$$

where $\lambda = \lambda_1(1-\lambda_2)+\lambda_2$, and it is very easy to verify that $1-\lambda = (1-\lambda_1)(1-\lambda_2)$. The proof is complete ∎

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